Probability and Information

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Introduction

Data frequency and probability

- We have considered frequent itemsets to infer association rules (i.e. discover knowledge) from a transactional database (TDB).

TID	Items
1	Bread, Milk
2	Bread, Diapers, Beer, Eggs
3	Milk, Diapers, Beer, Coke
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The support of itemset A in TDB is the fraction of transactions with A:

$$\mathrm{supp}(A) = \frac{\#\text{transactions}(A)}{\#\text{transactions}} = \frac{n(A)}{n}$$

where $\#$ means 'the number n of' (e.g. $\mathrm{supp}(\text{bread}) = 4/5$).

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Laws of large numbers

The frequency of observing event E in n independent and identically distributed (i.i.d.) experiments converges (in some sense) to the probability of E:

$$\frac{n(E)}{n} \to P(E) \quad \text{as} \quad n \to \infty$$

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1970-80 Information geometry (e.g. Chentsov, Amari).

Sources of uncertainty

Complexity : the number of possible states of a system in question can be too large (e.g. predict how a chess game can develop after 10 moves?)

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Randomness : the system may be random by nature, and thus the uncertainty is irreducible.

What is probability?

Definition (Probability of event E)

the measure $P(E)$ of certainty that event E will occur and ranging from $P(E) = 0$ (impossible) to $P(E) = 1$ (certain):

$$\text{(Impossible)} \qquad 0 \leq P(E) \leq 1 \qquad \text{(Certain)}$$

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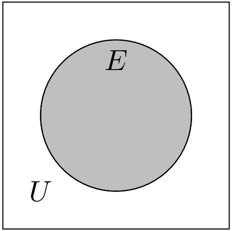
Example (Dice)

For a fair die, $P(6) = \frac{1}{6}$

Set-theoretic intuition

- Events E are considered as subsets of the universal set U:

$$E \subseteq U$$



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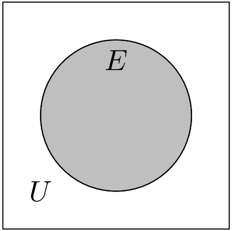

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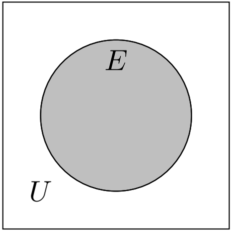
- Probability of E is a measure of a subset $E \subseteq U$.
- Probabilities of negation (not E), disjunction (A or B) and conjunction (A and B):

$$P(\bar{E}) = P(U - E)\,, \quad P(A \cup B)\,, \quad P(A \cap B)$$



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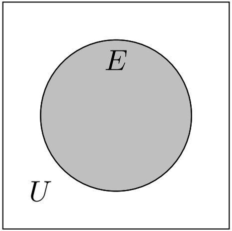


Universal set

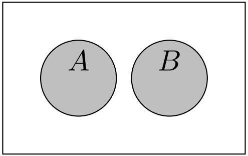
Because the unvierse is certain, we set

$$P(U) = 1$$

Additivity of probabilities

- For disjoint events $A \cap B = \varnothing$:

$$P(A \text{ or } B) = P(A) + P(B)$$



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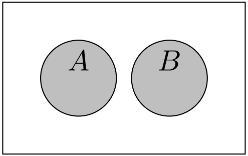

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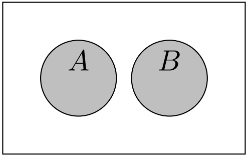

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Example

For a fair coin and a fair dice we have

$$\frac{1}{2} + \frac{1}{2} = 1 \qquad \frac{1}{6} + \frac{1}{6} + \frac{1}{6} + \frac{1}{6} + \frac{1}{6} + \frac{1}{6} = 1$$

Probability of negation

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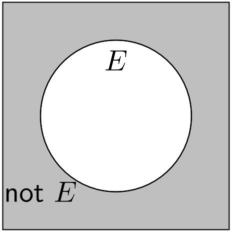

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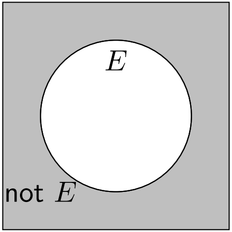

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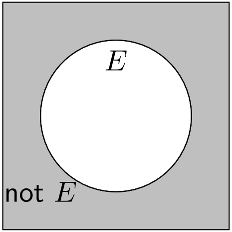

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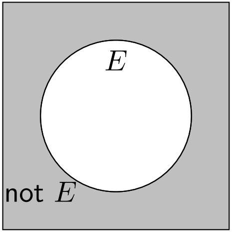
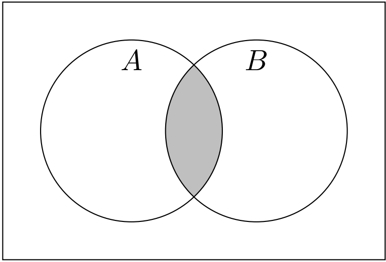


Empty set

$$P(\varnothing) = P(\text{not}U) = 1 - P(U) = 0$$

Joint probability

- Co-occurrence of events A and B together (e.g. clouds and rain) is their set intersection: $A \cap B$.



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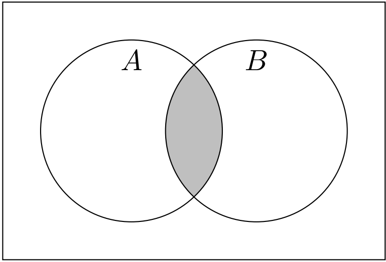
- Probability of $A \cap B$ is called joint probability:

$$P(A \cap B)$$



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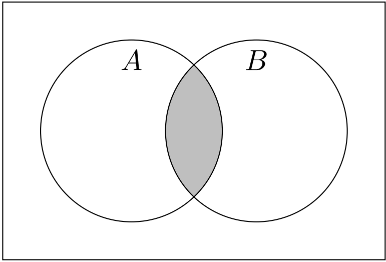

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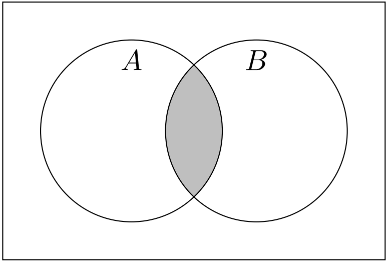

Example (Two coins)

A	B
heads	heads
heads	tails
tails	heads
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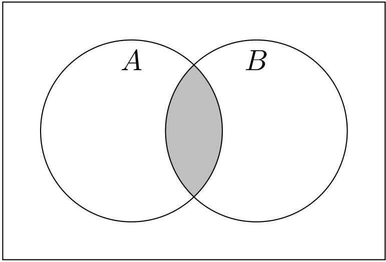

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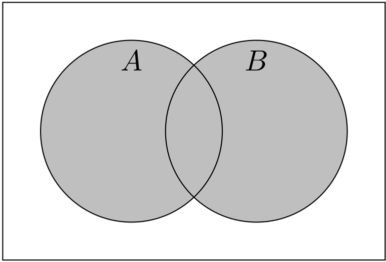
Probability of union

- Probability of A or B is:

  $$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$



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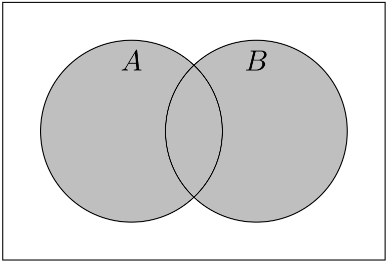

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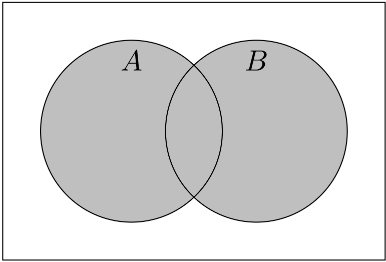

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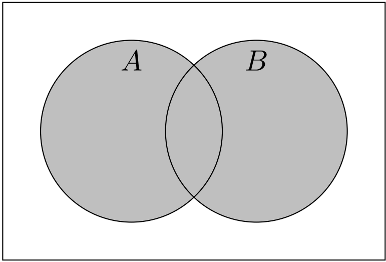

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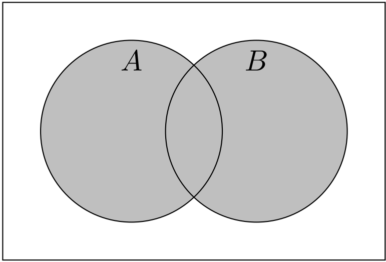

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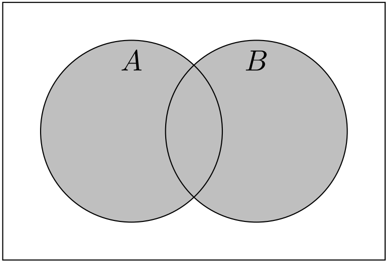

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- What is $P(\text{bread} \cup \text{milk})$?
- Using $P(\text{bread} \cap \text{milk}) = 3/5$ we have

$$P(\text{bread} \cup \text{milk}) = \frac{4}{5} + \frac{4}{5} - \frac{3}{5} = 1$$

Probability distributions

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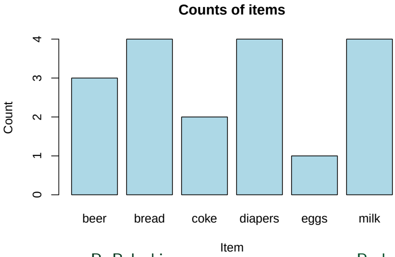

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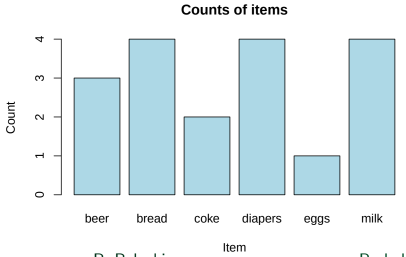

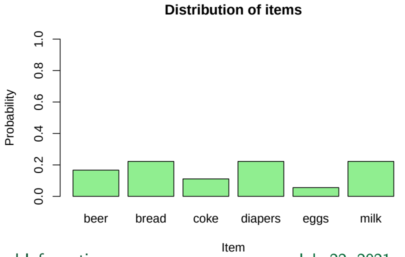

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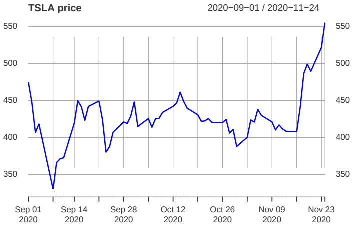
Distribution of Tesla prices



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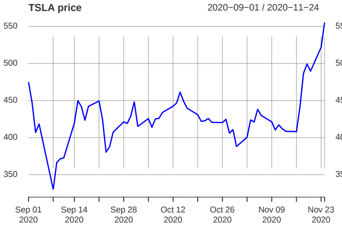

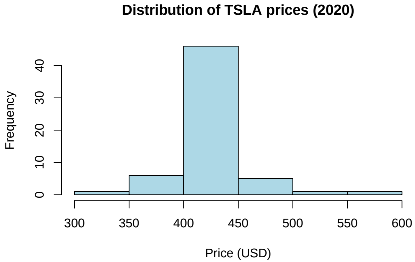

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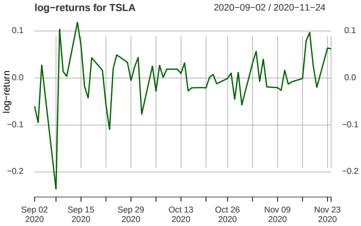

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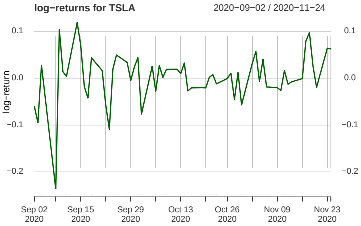

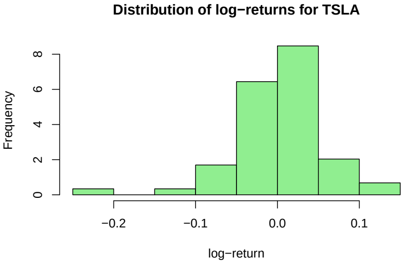

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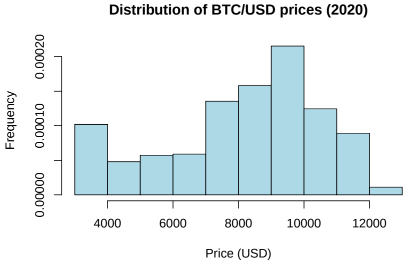

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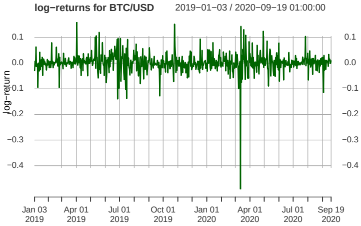

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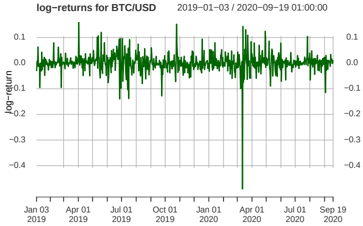

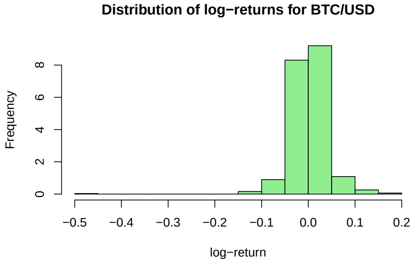

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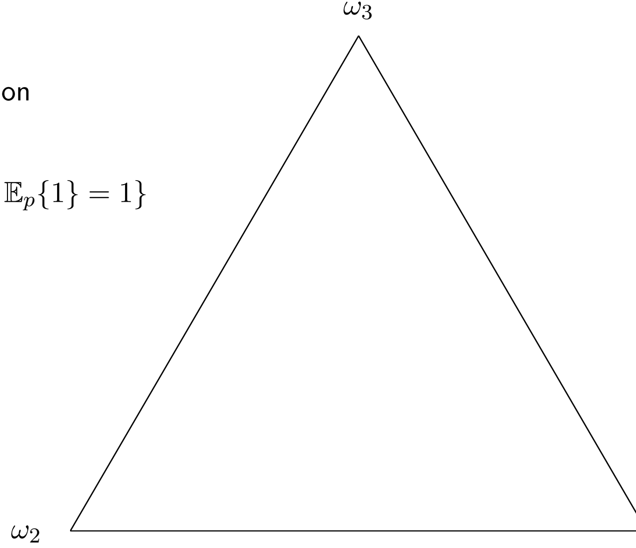

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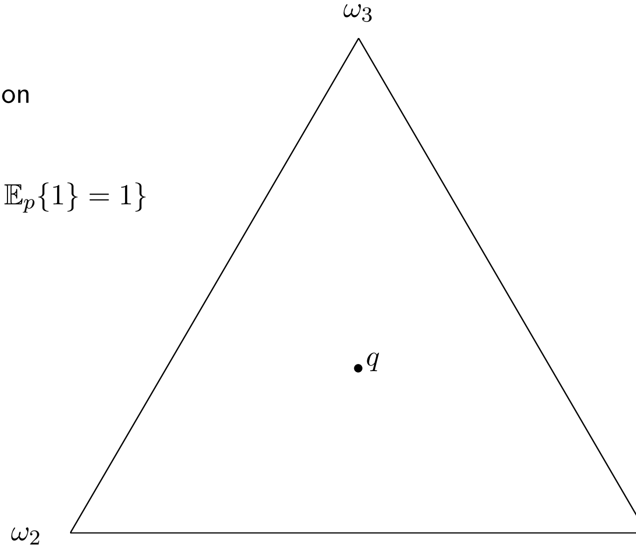

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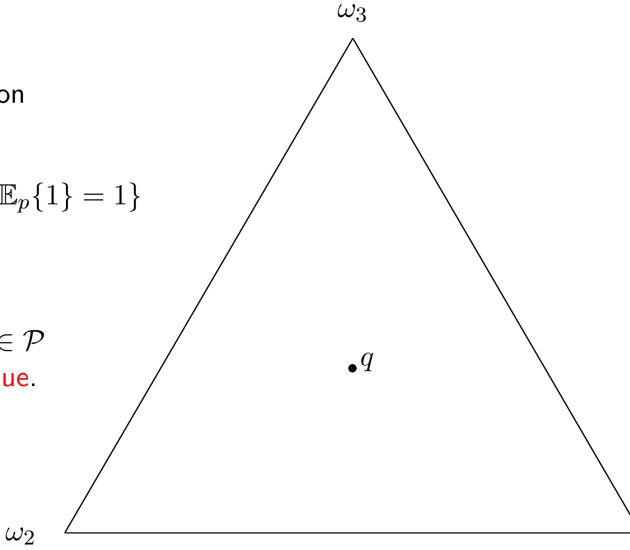

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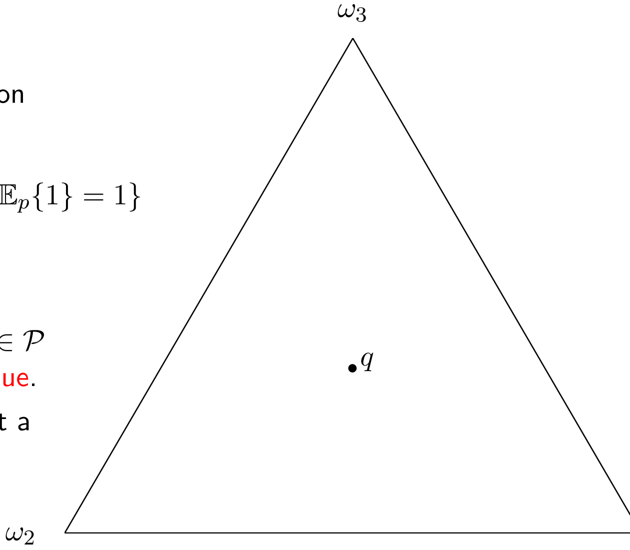

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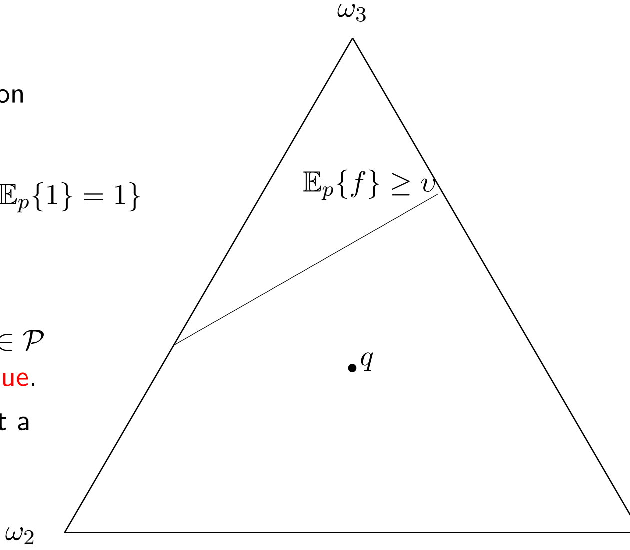

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Example

Let $\texttt{Age} = \{21, 18, 50, 23, 40\}$ and $P(\texttt{Age}) = \frac{1}{5}$. Then the mean age is

$$E\{\texttt{Age}\} = \frac{21 + 18 + 50 + 23 + 40}{5} = 30,4$$

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- Its square root is called standard deviation: $\mathrm{Sd}\{x\} = \sqrt{\mathrm{Var}\{x\}}$

Distribution vs Moments

- $P(\omega)$ gives us all moments of $x : \mathcal{A} \subseteq 2^\Omega \to \mathbb{R}$:

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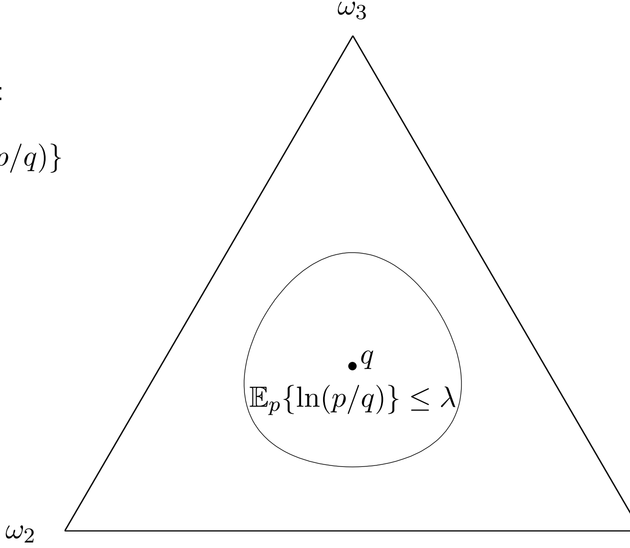
- What is better: To know $P(x)$ or to know moments $\mathbb{E}_P\{x^n\}$?

KL-divergence and $\Gamma(u) = \ln \Theta(u)$

- The KL-divergence between p, $q \in \mathcal{P}(\Omega)$:

  $D_{KL}[p, q] := \mathbb{E}_P\{\ln(p/q)\}$



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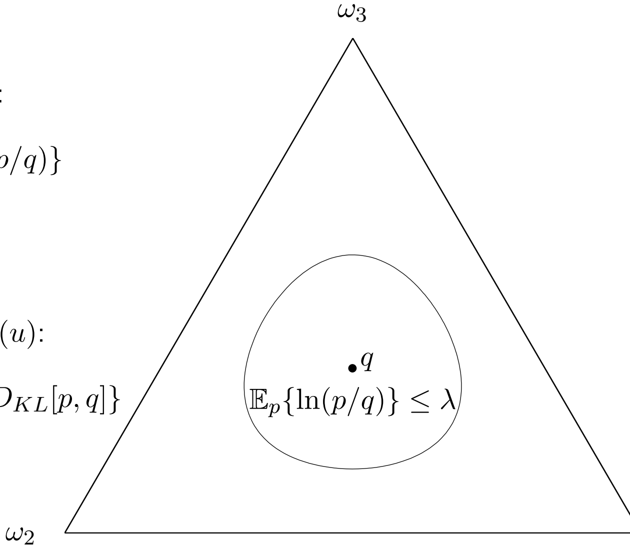
  $$\Gamma[u] = \sup_p \{\mathbb{E}_p\{u\} - D_{KL}[p, q]\}$$



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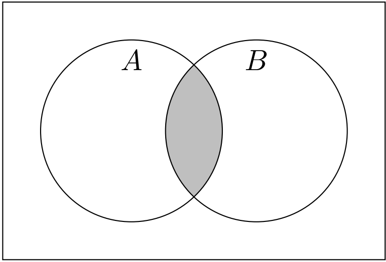
- Co-occurrence of events A and B together (e.g. clouds and rain) is their set intersection: $A \cap B$.



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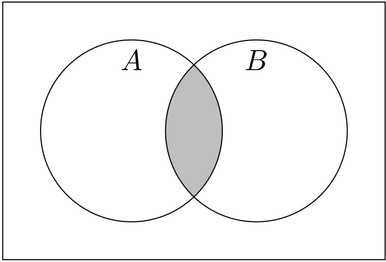

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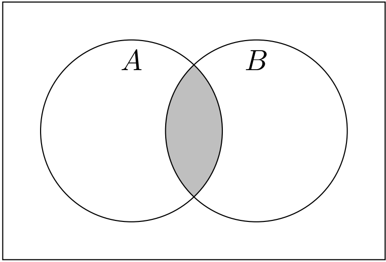

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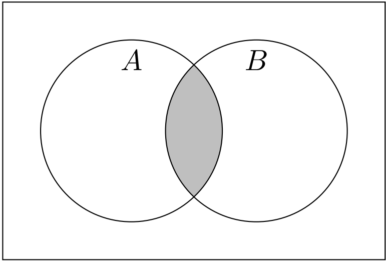

Example (Two independent fair coins)

Coin $A = \{\text{head}, \text{tail}\}$
Coin $B = \{\text{head}, \text{tail}\}$

$P(A \cap B) =$

	head	tail
head	1/4	1/4
tail	1/4	1/4

Marginal probabilities

Probabilities $P(A)$ or $P(B)$ are sometimes called marginal, because they can be obtained from joint probability $P(A \cap B)$ by summation:

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	no rain	rain
no clouds	$1/2$	0
clouds	$3/10$	$1/5$

$P(A) =$

no clouds	$1/2$
clouds	$1/2$

$P(B) =$

	no rain	rain
	$4/5$	$1/5$

Conditional probability

Question

How likely is it to rain if you see clouds?

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Definition (Conditional probability)

- The probability of event A conditioned on the outcome of B:

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Example (Clouds and rain)

- For $A = \{\text{clouds}, \text{clear sky}\}$ and $B = \{\text{rain}, \text{no rain}\}$, we can consider

$$P(\text{rain} \mid \text{clouds})$$

- Is it the same as $P(\text{rain})$?

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- Otherwise, if

$$P(A \mid B) \neq P(A) \qquad \text{or} \qquad P(B \mid A) \neq P(B)$$

events A and B are said to be statistically dependent.

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events A and B are said to be statistically dependent.

Example (Clouds and rain)

If you believe that rain is not possible without clouds, then

$$P(\text{clouds} \mid \text{rain}) = 1$$

Independence

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- From the decomposition of joint probability $P(A \cap B)$:

$$P(A \cap B) = P(A \mid B)P(B) = P(B \mid A)P(A)$$

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Example

In the TDB example, we saw

$$P(\text{Milk} \mid \text{Bread}) = \frac{3}{4} = \frac{3/5}{4/5}$$

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$$P(\text{rain} \mid \text{clouds}) = \frac{1 \times 1/5}{1/2} = \frac{2}{5}$$

Another view on independence

- The joint probability $P(A \cap B)$ (i.e. probability of A and B)

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Example (Two independent fair coins)

$$P(A \cap B) = \left[\begin{array}{r|cc} & \text{head} & \text{tail} \\ \hline \text{head} & 1/4 & 1/4 \\ \text{tail} & 1/4 & 1/4 \end{array}\right] \quad P(A) = \left[\begin{array}{r|c} & \\ \hline \text{head} & 1/2 \\ \text{tail} & 1/2 \end{array}\right]$$

$$P(B) = \left[\begin{array}{r|cc} & \text{head} & \text{tail} \\ \hline & 1/2 & 1/2 \end{array}\right] \quad P(\text{head}, \text{tail}) = \frac{1}{2} \times \frac{1}{2} = \frac{1}{4}$$

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$$P(\text{no clouds}, \text{rain}) = 0 \neq \frac{1}{2} \times \frac{1}{5}$$

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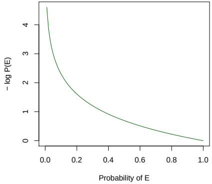

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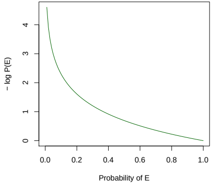

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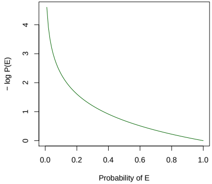

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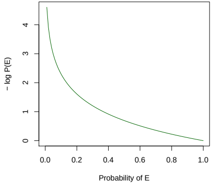

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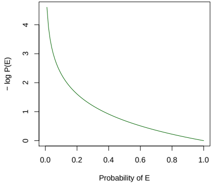

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- It is usually thought of as a measure of uncertainty, but it is also a measure of potential information.

Mutual information

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- Thus, information is the amount by which uncertainty is reduced.